

载流导线激励弹性约束轴向变速运动梁主-参联合共振*

PRINCIPAL-PARAMETRIC JOINT RESONANCE OF AN AXIALLY VARIABLE-SPEED MOVING BEAM WITH ELASTIC RESTRAINT EXCITED BY CURRENT-CARRYING WIRES

李晓靓**^{1,2} 胡宇达***^{1,2}

(1. 燕山大学 建筑工程与力学学院, 秦皇岛 066004)

(2. 燕山大学 河北省重型装备与大型结构力学可靠性重点实验室, 秦皇岛 066004)

LI Xiaojing^{1,2} HU Yuda^{1,2}

(1. School of Civil Engineering and Mechanics, Yanshan University, Qinhuangdao 066004, China)

(2. Hebei Key Laboratory of Mechanical Reliability for Heavy Equipment and Large Structures, Yanshan University, Qinhuangdao 066004, China)

摘要 研究了两端弹性约束轴向变速运动梁的主-参联合共振问题。根据哈密顿原理,推导电梁在平行载流导线激发磁场中的磁弹性非线性振动方程。由两端弹性约束边界条件得到振型函数,再对方程进行伽辽金积分法离散,然后利用多尺度法得到稳态运动的近似解析解,并对稳态解的稳定性进行了分析。通过算例,用数值方法对解析解结果进行验证,并分析不同参数对共振振幅及频率的影响。结果表明,系统发生主-参联合共振时,弹簧刚度和外激励力主要影响系统振幅,而轴向速度、电流、轴向力对振幅和共振频率都有影响;系统的稳定区域随弹簧刚度的增大而增加。

关键词 磁弹性 轴向变速运动梁 弹性约束 主-参联合共振 多尺度法

中图分类号 O322 O343.5

Abstract The principal-parametric joint resonance of an axially variable-speed moving beam with elastic restraint at both ends was studied. Based on Hamilton principle, the magneto-elastic nonlinear vibration equation of the beam was derived, in which magnetic field generated by parallel current-carrying wires was introduced in modelling. The mode shape function was obtained by means of elastic boundary conditions, and then the Galerkin method was employed to separate time-items and space-items from each other. The approximate analytical solution and the stability of steady-state motion were achieved by multi-scale method. In analyzing the computational results, the reliability of the analytical solutions were verified by numerical calculation results corresponding to parameters. In addition, the effects of different parameters on the resonance amplitude and frequency were analyzed. The results show that the stiffnesses of springs and the external excitation force may only affect the amplitude of the system, and the axial velocity, the current and the axial force may lead to the variation of the amplitude and frequency of the resonance, where the stability region increases with the increase of the stiffness of springs.

Key words Magneto-elastic; Axially variable-speed moving beam; Elastic restraint; Principal-parametric joint resonance; Multiple-scale method

Corresponding author: HU Yuda, E-mail: huyuda03@163.com, Tel: +86-335-8057101, Fax: +86-335-8057101

The project supported by the National Natural Science Foundation of China (No. 12172321), and the Natural Science Foundation of Hebei Province (No. A2020203007).

Manuscript received 20221129, in revised form 20230112.

0 引言

轴向运动梁在各个领域都有着广泛的应用,如航

空航天部件、高速机械部件、可伸缩机翼、带锯片、磁带、机械臂等许多运动结构部件。虽然学者们对轴向运动梁结构采用不同的解析或数值方法进行了一定的

* 20221129 收到初稿, 20230112 收到修改稿。国家自然科学基金项目(12172321), 河北省自然科学基金项目(A2020203007)资助。

** 李晓靓, 女, 1980 年生, 河北保定人, 汉族, 燕山大学在读博士研究生, 主要研究方向为非线性振动。

*** 胡宇达(通信作者), 男, 1968 年生, 黑龙江东宁人, 汉族, 燕山大学教授, 博士研究生导师, 主要研究方向为非线性振动和磁弹性力学。

研究,但这些研究大多将实际的复杂边界条件简化成经典的边界条件,如简支、固支、自由边界等,但是弹性约束模型能较好地模拟实际情况。在工程实际中,许多大型机器设备通过铺设弹性地基、使用弹性支座来减轻结构振动。支撑在弹性地基基础上的梁,或者超长型隔振台座,还有桥梁工程中广泛使用的弹性支座上的梁等常会简化为弹性支撑梁模型进行研究。目前,已有学者对在复杂约束边界下梁结构的振动分析进行了建模,取得了初步的研究成果。赵雨皓等^[1]采用改进傅里叶级数展开建立了轴向载荷条件下弹性边界约束梁结构振动分析模型。滕兆春等^[2]基于 Timoshenko 梁理论研究弹性地基上转动功能梯度材料梁的自由振动。YAU J D 等^[3]采用解析方法研究了带有弹性支座的桥梁在移动列车荷载作用下的动力响应。鲍四元等^[4]基于地基梁的拉格朗日函数,利用瑞利-里兹方法把振动问题转化为矩阵特征值问题,研究端部弹性约束地基梁的自由振动特性。LIU Y^[5]对弹性支撑梁的受迫振动展开了分析,得到了在移动载荷下,移动速度与支撑刚度对系统响应的影响。工程应用中较多结构的轴向力和轴向速度并非恒定,加上外界激励的周期性变化,使得轴向运动梁有可能发生动态失稳,形成参激共振。为了有效降低梁结构在外部动态载荷激励下的不利振动影响,弹性约束梁结构的参激振动研究长期以来广受关注。张弛等^[6]主要研究了受轴向激励弹性支撑梁的自由振动特性、参激振动稳定性和非线性稳态响应。张登博等^[7]引入由黏弹性引起的非齐次边界条件,同时引入轴向加速度引起的径向变化张力,采用直接多尺度法分析轴向变速运动黏弹性 Euler 梁的次谐波参数共振。李威等^[8]采用离散奇异卷积法对弹性约束梁在动轴向载荷作用下的参数失稳问题进行了研究。ZHANG D B 等^[9]对具有 1:3 内共振的不规则边界条件的轴向加速梁进行了解析分析和数值研究。陈明飞等^[10]采用人工弹簧模拟曲梁边界实现任意边界约束,对任意曲率半径的功能梯度曲梁进行几何建模和振动特性分析。CHEN L Q 等^[11]研究了弹性边界约束的轴向变速梁的参数共振问题。

随着人们对电、磁与介质之间的相互作用机制的认识不断深入,导电梁的运行环境越来越复杂,电磁场环境中工作的轴向运动梁表现出复杂的耦合动力学行为,往往出现振动失稳、高频噪声以及设备故障,或将直接影响系统的安全性与可靠性。导电弹性结构在多物理场耦合作用下的动力学研究已成为引人关注的研究领域。HU Y D 等^[12] 683-695^[13-14]对轴向运动梁在磁场环境下的非线性共振问题进行了探索,研究了轴向运动梁的磁弹性非线性主-内联合共振及相关的稳定性

分析。NAYAK B 等^[15]阐述了磁流变弹性体的基础梁的导电表面在不同边界条件下的动态分析。PRATHIHER B 等^[16]对变磁场中受轴向周期载荷作用下悬臂梁的主参数振动及其稳定性进行了分析。郑晓静等^[17]研究了铁磁导电梁式板在横向均匀磁场中的动力特性。WANG L 等^[18]针对轴向运动悬臂梁在磁场力作用下的振动控制问题进行了分析。LIU M F 等^[19]考虑横向磁场、轴向载荷和外力之间的相互作用,研究了具有一般边界条件的磁弹性梁的振动问题。

综上所述,对于轴向运动梁的非线性振动问题,大多是对经典边界梁的研究。针对弹性约束梁的振动问题的研究大多是对结构的自由振动或受迫振动的研究。因此,多物理场耦合作用下弹性约束轴向运动梁的联合共振问题的研究具有理论和实际意义。本文考虑梁的振动引起梁与导线之间的距离发生变化从而引起磁场的变化,同时考虑时变横向载荷及时变轴向载荷作用,推导梁的非线性磁弹性振动方程,分析弹簧刚度、电流、轴向速度、轴向力、外激励力等参数对主-参联合共振的影响。

1 轴向变速运动梁控制方程

图 1 所示为两端弹性约束梁。支撑弹簧刚度分别为 k_l 和 k_r , 沿形心轴 x 方向以速度 $c = c_0 + c_1 \cos(\omega_1 t)$ 运动,受载流导线激发的磁场作用,并受轴向载荷 $F_{0x} = F_0 + F_1 \cos(\omega_2 t)$ 及均布横向载荷 $P_z = P_0 \cos(\omega_3 t)$ 作用, c_0 、 F_0 、 P_0 分别为恒定轴向速度、恒定轴向力、外激励力的幅值, c_1 、 F_1 分别为时变轴向速度幅值、时变轴向力幅值, ω_1 、 ω_2 、 ω_3 分别为轴向速度、轴向力、外激励力的频率。梁长为 l , 高为 h , 宽为 b , 截面积 $A = bh$ 。

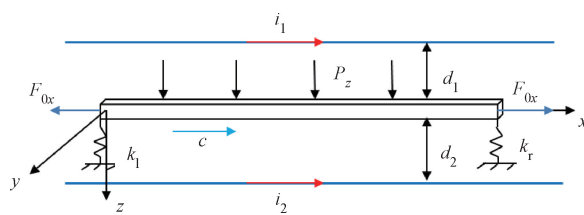


图 1 平行载流导线间弹性约束梁模型

Fig. 1 Model of the beam with elastic constraints between parallel current-carrying wires

考虑轴向速度和横向速度,轴向运动梁的总动能为

$$T = \frac{\rho A}{2} \int_0^l \left[c^2 + \left(\frac{\partial w}{\partial t} + c \frac{\partial w}{\partial x} \right)^2 \right] dx \quad (1)$$

式中, ρ 为密度; w 为横向位移, $w = w(x, t)$; t 为时间变量。

梁的势能可以表示为

$$U = \int_0^l [F_{0x} \varepsilon_x + \frac{1}{2} EI (\frac{\partial^2 w}{\partial x^2})^2 + \frac{1}{2} EA \varepsilon_x^2] dx \quad (2)$$

式中, ε_x 为轴向应变, $\varepsilon_x = \frac{1}{2} (\frac{\partial w}{\partial x})^2$; I 为截面惯性矩,

$$I = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dy dz; E \text{ 为梁的弹性模量。}$$

考虑导电梁不通电的情况, 平行导线通入直流电且电流方向相同。在梁处激发的磁感应强度 $B_0 = [0, B_{0y}, 0]$, 利用毕奥-萨伐尔电磁定律^[20]可知

$$B_{0y} = \frac{\mu_0 i_1}{2\pi(d_1 - w)} - \frac{\mu_0 i_2}{2\pi(d_2 + w)} \quad (3)$$

式中, i_1, i_2 为两导线的电流值; d_1, d_2 为两导线与梁之间的距离; μ_0 为真空磁导率。

受电磁场作用, 导电梁单位长度上的洛伦兹力分量为^{[12] 683-695}

$$F_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} J_x B_{0y} dy dz = A(-\sigma_0 B_{0y} \frac{\partial w}{\partial t} - \sigma_0 B_{0y} c \frac{\partial w}{\partial x}) \quad (4)$$

式中, σ_0 为电导率; J_x 为磁场在梁内感应的电流密度矢量沿 x 轴方向分量。

将式(3)中的电流值项用泰勒级数展开, 略去三次以上的高阶项, 结合上面给出的各表达式, 基于哈密

顿变分原理, 得到磁场中轴向变速运动导电梁关于 w 的偏微分控制方程为

$$\rho A \frac{\partial^2 w}{\partial t^2} + 2\rho A c \frac{\partial^2 w}{\partial x \partial t} + \rho A c^2 \frac{\partial^2 w}{\partial x^2} + \rho A \frac{\partial c}{\partial t} \frac{\partial w}{\partial x} - F_{0x} \frac{\partial^2 w}{\partial x^2} - \frac{3}{2} EA (\frac{\partial w}{\partial x})^2 \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} + \frac{A \sigma_0 \mu_0^2}{4\pi^2} (\frac{\partial w}{\partial t} + c \frac{\partial w}{\partial x}) (\alpha_1^2 + \alpha_2 w + \alpha_3 w^2) - P_z = 0 \quad (5)$$

式中, $\alpha_1 = \frac{i_1}{d_1} - \frac{i_2}{d_2}$; $\alpha_2 = \frac{2i_1^2}{d_1^3} - \frac{2i_2^2}{d_2^3} - \frac{2i_1 i_2}{d_1 d_2} (\frac{1}{d_1} - \frac{1}{d_2})$;

$$\alpha_3 = \frac{3i_1^2}{d_1^4} + \frac{3i_2^2}{d_2^4} - \frac{2i_1 i_2}{d_1 d_2} (\frac{1}{d_1^2} - \frac{1}{d_1 d_2} + \frac{1}{d_2^2})。$$

2 弹性约束下方程的离散化

两端弹性约束梁的边界条件为

$$\begin{cases} x=0, & EI \frac{\partial^2 w}{\partial x^2} = 0, EI \frac{\partial^3 w}{\partial x^3} + k_1 w = 0 \\ x=l, & EI \frac{\partial^2 w}{\partial x^2} = 0, EI \frac{\partial^3 w}{\partial x^3} - k_1 w = 0 \end{cases} \quad (6)$$

设满足边界条件式(6)的位移解为

$$w = q_1(t) X_1 \quad (7)$$

式中, $X_1 = C_1 \cos p_1 x + C_2 \sin p_1 x + C_3 \cosh p_1 x + C_4 \sinh p_1 x$ 。

将式(7)代入式(6)可得方程组:

$$\begin{bmatrix} -p_1^2 EI & 0 & p_1^2 EI & 0 \\ -k_1 & p_1^3 EI & -k_1 & p_1^3 EI \\ -p_1^2 EI \cos p_1 l & -p_1^2 EI \sin p_1 l & p_1^2 EI \cosh p_1 l & p_1^2 EI \sinh p_1 l \\ p_1^3 EI \sin p_1 l - k_1 \cos p_1 l & -p_1^3 EI \cos p_1 l - k_1 \sin p_1 l & p_1^3 EI \sinh p_1 l - k_1 \cosh p_1 l & p_1^3 EI \cosh p_1 l - k_1 \sinh p_1 l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

在实际工程中, 梁两端的支撑情况往往是相同的, 这里取 $k_1 = k_r = k$ 。方程组存在非零解, 则系数矩阵的行列式为零, 由此求得系统的固有频率。本文用数值方法求得不同 k 值对应的 p_1 值。继而可以得到对应的振型函数系数 $C_1 \sim C_4$, 其表达式为

$$\begin{aligned} C_1 &= [2k^2 \sin(p_1 l) \sinh(p_1 l) + E^2 l^2 p_1^6 \cosh(p_1 l)^2 - \\ &E^2 l^2 p_1^6 \sinh(p_1 l)^2 - E^2 l^2 p_1^6 \cos(p_1 l) \cosh(p_1 l) + \\ &E^2 l^2 p_1^6 \sin(p_1 l) \sinh(p_1 l) + Elp_1^3 k \cos(p_1 l) \sinh(p_1 l) - \\ &3Elp_1^3 k \cosh(p_1 l) \sin(p_1 l)] / [2k^2 \cosh(p_1 l) \sin(p_1 l) + \\ &Elp_1^3 k \cos(p_1 l)^2 + Elp_1^3 k \sin(p_1 l)^2 - E^2 l^2 p_1^6 \cos(p_1 l) \cdot \\ &\sinh(p_1 l) - E^2 l^2 p_1^6 \cosh(p_1 l) \sin(p_1 l) + 3Elp_1^3 k \cdot \\ &\cos(p_1 l) \cosh(p_1 l) - Elp_1^3 k \sin(p_1 l) \sinh(p_1 l)] \\ C_2 &= -[2k^2 \cos(p_1 l) \sinh(p_1 l) - Elp_1^3 k \cosh(p_1 l)^2 + \\ &Elp_1^3 k \sinh(p_1 l)^2 + E^2 l^2 p_1^6 \cos(p_1 l) \sinh(p_1 l) + \\ &E^2 l^2 p_1^6 \cosh(p_1 l) \sin(p_1 l) - 3Elp_1^3 k \cos(p_1 l) \cosh(p_1 l) - \end{aligned}$$

$$\begin{aligned} &Elp_1^3 k \sin(p_1 l) \sinh(p_1 l)] / [2k^2 \cosh(p_1 l) \sin(p_1 l) + \\ &Elp_1^3 k \cos(p_1 l)^2 + Elp_1^3 k \sin(p_1 l)^2 - E^2 l^2 p_1^6 \cos(p_1 l) \cdot \\ &\sinh(p_1 l) - E^2 l^2 p_1^6 \cosh(p_1 l) \sin(p_1 l) + \\ &3Elp_1^3 k \cos(p_1 l) \cosh(p_1 l) - Elp_1^3 k \sin(p_1 l) \cdot \\ &\sinh(p_1 l)] \\ C_3 &= -[2k^2 \sin(p_1 l) \sinh(p_1 l) + E^2 l^2 p_1^6 \cos(p_1 l)^2 + \\ &E^2 l^2 p_1^6 \sin(p_1 l)^2 - E^2 l^2 p_1^6 \cos(p_1 l) \cosh(p_1 l) - \\ &E^2 l^2 p_1^6 \sin(p_1 l) \sinh(p_1 l) + 3Elp_1^3 k \cos(p_1 l) \cdot \\ &\sinh(p_1 l) - Elp_1^3 k \cosh(p_1 l) \sin(p_1 l)] / [2k^2 \cdot \\ &\cosh(p_1 l) \sin(p_1 l) + Elp_1^3 k \cos(p_1 l)^2 + Elp_1^3 k \sin(p_1 l)^2 - \\ &E^2 l^2 p_1^6 \cos(p_1 l) \sinh(p_1 l) - E^2 l^2 p_1^6 \cosh(p_1 l) \cdot \\ &\sin(p_1 l) + 3Elp_1^3 k \cos(p_1 l) \cosh(p_1 l) - Elp_1^3 k \cdot \\ &k \sin(p_1 l) \sinh(p_1 l)] \\ C_4 &= 1 \end{aligned}$$

将式(7)代入式(5)中,采用伽辽金积分法,得到磁场中轴向运动梁的横向振动微分方程,即

$$\begin{aligned} & \rho A D_{11} \ddot{q}_1(t) + 2\rho A(c_0 + c_1 \cos \omega_1 t) C_{11} \dot{q}_1(t) + \\ & [\rho A(c_0 + c_1 \cos \omega_1 t)^2 - (F_0 + F_1 \cos \omega_2 t)] B_{11} q_1(t) + \\ & E I A_{11} q_1(t) - \frac{3}{2} E A S_{11} q_1^3(t) + \frac{A \sigma_0 \mu_0^2}{4\pi^2} \{ \alpha_1^2 D_{11} \dot{q}_1(t) + \\ & \alpha_1^2 (c_0 + c_1 \cos \omega_1 t) C_{11} q_1(t) + \alpha_2 H_{11} \dot{q}_1(t) q_1(t) + \\ & \alpha_2 (c_0 + c_1 \cos \omega_1 t) K_{11} q_1^2(t) + \alpha_3 M_{11} \dot{q}_1(t) q_1^2(t) + \\ & \alpha_3 (c_0 + c_1 \cos \omega_1 t) N_{11} q_1^3(t) \} - \rho A c_1 \omega_1 \sin \omega_1 t C_{11} q_1(t) - \\ & P_0 \cos \omega_3 t E_1 = 0 \end{aligned} \quad (9)$$

$$\text{式中, } A_{11} = \int_0^l \frac{d^4 X_1}{dx^4} X_1 dx; B_{11} = \int_0^l \frac{d^2 X_1}{dx^2} X_1 dx; C_{11} =$$

$$\begin{aligned} & \ddot{q}_1(\tau) + g_1^2 q_1(\tau) = -\varepsilon [(\delta_{111} \cos \Omega_1 \tau + \delta_{211} \cos 2\Omega_1 \tau + \delta_{311} \cos \Omega_2 \tau + \delta_{411} \Omega_1 \sin \Omega_1 \tau) q_1(\tau) + (\mu_{11} + \\ & \mu'_{11} \cos \Omega_1 \tau) \dot{q}_1(\tau) + (\chi_{11} + \chi'_{11} \cos \Omega_1 \tau) q_1^2(\tau) + (n_{11} + n'_{11} \cos \Omega_1 \tau) q_1^3(\tau) + h_{11} \dot{q}_1(\tau) q_1(\tau) + \\ & m_{11} \dot{q}_1(\tau) q_1^2(\tau) - f_1 \cos \Omega_3 \tau] \end{aligned} \quad (10)$$

$$\begin{aligned} \text{式中, } q_1(\tau) &= \frac{q_1(t)}{h}; \omega_0 = p_1^2 \sqrt{\frac{EI}{\rho A}}; \tau = \omega_0 t; \eta = \frac{c_0}{\omega_0}; \zeta = \frac{F_0}{\rho A \omega_0^2}; \mu'_{11} = \frac{2c_1 C_{11}}{\omega_0 D_{11}}; \Omega_1 = \frac{\omega_1}{\omega_0}; \Omega_2 = \frac{\omega_2}{\omega_0}; \Omega_3 = \frac{\omega_3}{\omega_0}; \mu_{11} = \frac{\sigma_0 \mu_0^2 \alpha_1^2 D_{11}}{4\pi^2 \rho \omega_0 D_{11}} + \frac{2c_0 C_{11}}{\omega_0 D_{11}}; \\ g_1^2 &= \frac{\eta^2 B_{11}}{D_{11}} - \frac{\zeta B_{11}}{D_{11}} + \frac{E I A_{11}}{\rho A \omega_0^2 D_{11}} + \frac{\mu_0^2 \sigma_0 \alpha_1^2 c_0 C_{11}}{4\pi^2 \rho \omega_0^2 D_{11}} + \frac{c_1^2 B_{11}}{2\omega_0^2 D_{11}}; \delta_{111} = \frac{\mu_0^2 \sigma_0 \alpha_1^2 c_1 C_{11}}{4\pi^2 \rho \omega_0^2 D_{11}} + \frac{2c_0 c_1 B_{11}}{\omega_0^2 D_{11}}; \delta_{211} = \frac{c_1^2 B_{11}}{2\omega_0^2 D_{11}}; \delta_{311} = \frac{-F_1 B_{11}}{\rho A \omega_0^2 D_{11}}; \\ \chi_{11} &= \frac{\mu_0^2 \sigma_0 \alpha_2 c_0 h K_{11}}{4\pi^2 \rho \omega_0^2 D_{11}}; \chi'_{11} = \frac{\mu_0^2 \sigma_0 \alpha_2 c_1 h K_{11}}{4\pi^2 \rho \omega_0^2 D_{11}}; m_{11} = \frac{\mu_0^2 \sigma_0 \alpha_3 M_{11} h^2}{4\pi^2 \rho D_{11} \omega_0}; n_{11} = \frac{\mu_0^2 \sigma_0 \alpha_3 c_0 h^2 N_{11}}{4\pi^2 \rho \omega_0^2 D_{11}} - \frac{3 E S_{11} h^2}{2 \rho \omega_0^2 D_{11}}; n'_{11} = \frac{\mu_0^2 \sigma_0 \alpha_3 c_1 h^2 N_{11}}{4\pi^2 \rho \omega_0^2 D_{11}}; \\ h_{11} &= \frac{\mu_0^2 \sigma_0 \alpha_2 H_{11} h}{4\pi^2 \rho D_{11} \omega_0}; \delta_{411} = \frac{-c_1 C_{11}}{\omega_0 D_{11}}; f_1 = \frac{P_0 E_1}{D_{11} \rho \omega_0^2 h A}. \end{aligned}$$

引入不同的时间尺度 $T_0 = \tau, T_1 = \varepsilon \tau$, 则 1 阶近似解可表示为

$$q_1(\tau, \varepsilon) = q_{11}(T_0, T_1) + \varepsilon q_{12}(T_0, T_1) \quad (11)$$

将式(11)代入式(10)后展开,令 ε 的同次幂项系数相等,可得到:

关于 ε^0 的近似方程为

$$D_0^2 q_{11} + g_1^2 q_{11} = 0 \quad (12)$$

关于 ε^1 的近似方程为

$$\begin{aligned} D_0^2 q_{12} + g_1^2 q_{12} &= -[2D_0 D_1 q_{11} + (\delta_{111} \cos \Omega_1 \tau + \delta_{211} \cos 2\Omega_1 \tau + \delta_{311} \cos \Omega_2 \tau + \delta_{411} \Omega_1 \sin \Omega_1 \tau) q_{11}(\tau) + (\mu_{11} + \mu'_{11} \cdot \\ & \cos \Omega_1 \tau) D_0 q_{11} + (\chi_{11} + \chi'_{11} \cos \Omega_1 \tau) q_{11}^2(\tau) + (n_{11} + n'_{11} \cos \Omega_1 \tau) q_{11}^3(\tau) + h_{11} q_{11} D_0 q_{11} + m_{11} q_{11}^2 D_0 q_{11} - \\ & f_1 \cos \Omega_3 \tau] \end{aligned} \quad (13)$$

$$\text{式中, } D_0 = \frac{\partial}{\partial T_0}; D_1 = \frac{\partial}{\partial T_1}.$$

设式(12)的通解形式为

$$\begin{aligned} & \int_0^l \frac{dX_1}{dx} X_1 dx; D_{11} = \int_0^l X_1 X_1 dx; E_1 = \int_0^l X_1 dx; H_{11} = \int_0^l X_1^2 dx; \\ & K_{11} = \int_0^l \frac{dX_1}{dx} X_1 X_1 dx; M_{11} = \int_0^l X_1^3 X_1 dx; N_{11} = \int_0^l \frac{dX_1}{dx} X_1^2 X_1 dx; \\ & S_{11} = \int_0^l \frac{d^2 X_1}{dx^2} \left(\frac{dX_1}{dx} \right)^2 X_1 dx. \end{aligned}$$

3 多尺度方法求解主-参数联合共振问题

应用多尺度法^[21]求解系统的主-参数联合共振问题。对式(9)进行无量纲化处理,针对方程具有弱非线性特点及在算例中参数取值时考虑系统的弱非线性,同时为了便于解析求解,方程中引入小参数 ε 得到

$$q_{11} = A_1(T_1) e^{ig_1 T_0} + \bar{A}_1(T_1) e^{-ig_1 T_0} \quad (14)$$

式中, $i^2 = -1$; A_1 为未知复数; $\bar{A}_1$ 为 A_1 的共轭项。

将式(14)代入式(13)得到

$$\begin{aligned} D_0^2 q_{12} + g_1^2 q_{12} &= -\{ 2A'_1 g_1 i e^{ig_1 T_0} + \frac{1}{2} [\delta_{111} (A_1 e^{ig_1 T_0 + i\Omega_1 \tau} + \\ & \bar{A}_1 e^{-ig_1 T_0 + i\Omega_1 \tau}) + \delta_{211} (A_1 e^{ig_1 T_0 + i2\Omega_1 \tau} + \bar{A}_1 e^{-ig_1 T_0 + i2\Omega_1 \tau}) + \\ & \delta_{311} (A_1 e^{ig_1 T_0 + i\Omega_2 \tau} + \bar{A}_1 e^{-ig_1 T_0 + i\Omega_2 \tau}) + \delta_{411} i \Omega_1 \cdot \\ & (A_1 e^{ig_1 T_0 - i\Omega_1 \tau} + \bar{A}_1 e^{-ig_1 T_0 - i\Omega_1 \tau})] + [\mu_{11} (ig_1 A_1 e^{ig_1 T_0} + \\ & \frac{\mu'_{11}}{2} (ig_1 A_1 e^{ig_1 T_0 + i\Omega_1 \tau} - ig_1 \bar{A}_1 e^{-ig_1 T_0 + i\Omega_1 \tau})) + [\chi_{11} \cdot \\ & (A_1^2 e^{2ig_1 T_0} + \bar{A}_1 A_1) + \frac{\chi'_{11}}{2} (A_1^2 e^{2ig_1 T_0 + i\Omega_1 \tau} + A_1 \bar{A}_1 e^{i\Omega_1 \tau} + \\ & \bar{A}_1^2 e^{-2ig_1 T_0 + i\Omega_1 \tau} + A_1 \bar{A}_1 e^{i\Omega_1 \tau})] + n_{11} [A_1^3 e^{3ig_1 T_0} + \\ & 3A_1^2 \bar{A}_1 e^{ig_1 T_0} + \frac{n'_{11}}{2} [A_1^3 e^{3ig_1 T_0 + i\Omega_1 \tau} + 3A_1^2 \bar{A}_1 e^{ig_1 T_0 + i\Omega_1 \tau} + \end{aligned}$$

$$\begin{aligned} & \bar{A}_1^3 e^{-i3g_1 T_0 + i\Omega_1 \tau} + 3\bar{A}_1^2 A_1 e^{-ig_1 T_0 + i\Omega_1 \tau}] + \\ & h_{11} [(ig_1 A_1 A_1 e^{2ig_1 T_0} - ig_1 \bar{A}_1 A_1)] + \\ & m_{11} [(ig_1 A_1 A_1^2 e^{ig_1 T_0} + ig_1 A_1 A_1 \bar{A}_1 e^{ig_1 T_0} + \\ & ig_1 A_1 \bar{A}_1^2 e^{-ig_1 T_0} + ig_1 A_1 \bar{A}_1 A_1 e^{ig_1 T_0})] - \\ & \frac{f_1}{2} e^{i\Omega_3 \tau} \} + cc \end{aligned} \quad (15)$$

式中, cc 为其前面各项的共轭项。

3.1 主-参数联合共振

由式(15)可知, 系统存在多种不同组合形式的联合共振状态, 本文只选取一种情况进行研究, 即按式(16)引进频率调谐值 $\sigma_1, \sigma_2, \sigma_3$ 。

$$\Omega_1 = 2g_1 + \varepsilon\sigma_1; \Omega_2 = 2g_1 + \varepsilon\sigma_2; \Omega_3 = g_1 + \varepsilon\sigma_3 \quad (16)$$

为避免久期项出现, 必须令 A_1 满足

$$\begin{aligned} & (2A_1' + \mu_{11} A_1) g_1 i + 3n_{11} A_1^2 \bar{A}_1 + 2m_{11} A_1^2 \bar{A}_1 g_1 i + \\ & e^{i\sigma_1 T_1} \left(\frac{\delta_{111}}{2} \bar{A}_1 - \frac{\mu_{11}'}{2} g_1 \bar{A}_1 i + \frac{3n_{11}'}{2} \bar{A}_1^2 A_1 \right) + \\ & e^{i\sigma_2 T_1} \frac{\delta_{311}}{2} \bar{A}_1 - e^{i\sigma_3 T_1} \frac{f_1}{2} = 0 \end{aligned} \quad (17)$$

对式(17)求解时, 复函数 A_1 写成如下指数形式:

$$A_1(T_1) = \frac{1}{2} a_1(T_1) e^{i\beta_1(T_1)} \quad (18)$$

将式(18)代入式(17)后分离实部虚部, 并令 $\phi_1 = \sigma_1 T_1 - 2\beta_1, \phi_2 = \sigma_2 T_1 - 2\beta_1, \phi_3 = \sigma_3 T_1 - \beta_1$, 经分析可知, 当且仅当 ϕ_1, ϕ_2, ϕ_3 为常数时才存在稳态运动。即

$a_1' = 0, \phi_1' = \phi_2' = \phi_3' = 0$ 。因此, 只有当 $\beta_1' = \frac{\sigma_1}{2} = \frac{\sigma_2}{2} = \sigma_3 = \sigma$ 时才存在稳态运动, 此时 $\phi_1 = \phi_2 = 2\phi_3 = 2\phi_4$, 由此可得

$$a_1 g_1 \phi_4' = \sigma g_1 a_1 - \eta_1 - \gamma_2 \cos 2\phi_4 + \gamma_7 \sin 2\phi_4 - \gamma_3 \cos 2\phi_4 - \gamma_4 \cos \phi_4 \quad (19a)$$

$$g_1 a_1' = -\frac{1}{2} g_1 \mu_{11} a_1 - \gamma_1 - \gamma_2 \sin 2\phi_4 - \gamma_7 \cos 2\phi_4 - \gamma_3 \sin 2\phi_4 - \gamma_4 \sin \phi_4 \quad (19b)$$

式中, $\eta_1 = \frac{3}{8} n_{11} a_1^3; \gamma_1 = \frac{m_{11}}{4} a_1^3 g_1; \gamma_7 = -\frac{\mu_{11}'}{4} a_1 g_1; \gamma_2 = \left(\frac{\delta_{111}}{4} a_1 + \frac{3n_{11}'}{16} a_1^3 \right); \gamma_4 = -\frac{f_1}{2}; \gamma_3 = \frac{\delta_{311}}{4} a_1$ 。

为了便于求解式(19), 采取变量变换:

$$\Psi = a_1(T_1) \cos \phi_4(T_1) \quad (20a)$$

$$\Gamma = a_1(T_1) \sin \phi_4(T_1) \quad (20b)$$

由式(19)、式(20)及稳态运动条件, 最终可得幅

频响应方程:

$$\left[\frac{-\gamma_4(O_2 + O_3)}{O_2^2 - O_3^2 - O_1 O_0} \right]^2 + \left[\frac{-O_1 \gamma_4}{O_1 O_0 + O_3^2 - O_2^2} \right]^2 = a_1^2 \quad (21)$$

式中, $O_1 = (\frac{1}{2} \mu_{11} a_1 g_1 + \gamma_1 + \gamma_7)/a_1; O_2 = (\gamma_2 + \gamma_3)/a_1; O_3 = (\sigma a_1 g_1 - \eta_1)/a_1; O_0 = (\frac{1}{2} \mu_{11} a_1 g_1 + \gamma_1 - \gamma_7)/a_1$ 。

由式(11)、式(14)、式(16)、式(18)可得1阶近似解析解为

$$q_1(\tau) = a_1 \cos(\Omega_3 \tau - \phi_4) \quad (22)$$

式中, a_1, ϕ_4 由式(19)、式(21)决定。

3.2 稳定性分析

为了分析系统的稳定性, 通过研究式(19a)、式(19b)奇点的性态, 分析系统稳态运动的稳定性, 设

$$a_1 = a_{10} + a_{11}, \phi_4 = \phi_{40} + \phi_{41} \quad (23)$$

式中, a_{10}, ϕ_{40} 为稳态运动下的稳态解; a_{11}, ϕ_{41} 为小的扰动量。

将式(23)代入式(19), 并关于 a_{11}, ϕ_{41} 展开, 这里 a_{10}, ϕ_{40} 满足式(21), 保留 a_{11}, ϕ_{41} 线性项, 最终得到特征方程:

$$\begin{vmatrix} J_{11} - \lambda & J_{12} \\ J_{21} & J_{22} - \lambda \end{vmatrix} = 0 \quad (24)$$

式中,

$$J_{11} = \frac{1}{g_1} \left[-\frac{1}{2} g_1 \mu_{11} - \frac{3m_{11}}{4} a_{10}^2 g_1 - \left(\frac{\delta_{111}}{4} + \frac{9n_{11}'}{16} a_{10}^2 \right) \cdot \sin 2\phi_{40} - \frac{\mu_{11}'}{4} g_1 \cos 2\phi_{40} - \frac{\delta_{311}}{4} \sin 2\phi_{40} \right];$$

$$J_{12} = \frac{1}{g_1} \left(-2A_{20} \cos 2\phi_{40} + 2A_{70} \sin 2\phi_{40} - 2A_{30} \cos 2\phi_{40} - A_{40} \cos \phi_{40} \right);$$

$$J_{21} = \frac{1}{g_1} \left(-\frac{3}{4} n_{11} a_{10} - \frac{3n_{11}'}{8} a_{10} \cos 2\phi_{40} - \frac{f_1}{2} \frac{1}{a_{10}^2} \cos \phi_{40} \right);$$

$$J_{22} = \frac{1}{a_{10} g_1} \left(2A_{20} \sin 2\phi_{40} + 2A_{70} \cos 2\phi_{40} + 2A_{30} \sin 2\phi_{40} + A_{40} \sin \phi_{40} \right);$$

$$A_{20} = \frac{\delta_{111}}{4} a_{10} + \frac{3n_{11}'}{16} a_{10}^3; A_{30} = \frac{\delta_{311}}{4} a_{10};$$

$$A_{40} = -\frac{f_1}{2}; A_{70} = -\frac{\mu_{11}'}{4} a_{10} g_1$$

系统稳态运动的稳定性可以通过矩阵方程的特征

根来判定。当特征根全部具有负实部,则系统稳定;若其特征根具有正实部,则系统不稳定。

4 算例分析

以铜制梁为例进行分析,主要参数有:梁长 $l = 0.3 \text{ m}$, 矩形截面宽 $b = 0.02 \text{ m}$ 、高 $h = 0.01 \text{ m}$, 材料电导率 $\sigma_0 = 5.714 \times 10^7 (\Omega \cdot \text{m})^{-1}$, 密度 $\rho = 8\,920 \text{ kg/m}^3$, 弹性模量 $E = 108 \text{ GPa}$, 真空磁导率 $\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$ 。由式(21)计算系统主-参数联合共振的幅频响应。表1给出不同弹簧刚度对应的 p_1 值。图2~图6给出不同参数条件下振幅随频率调谐参数、外激励力、轴向速度、电流及距离的变化规律。图7选取具有代表性的稳态解进行稳定性分析,给出不同弹簧刚度对稳定性区域的影响。图8给出了数值解与解析解的结果对比。

4.1 不同弹簧支撑刚度 p_1 值

由式(8)的系数矩阵行列式为零,可求得系统的固有频率。本文用数值方法求得不同 k 值对应的 p_1 值,如表1所示。两端弹簧刚度 $k=0$ 时,相当于自由-自由梁; $k=10^9 \text{ N/m}$ 时,可看成弹簧刚度无穷大,相当于简支-简支梁。针对这两种边界条件梁的 p_1 值与文献[22]进行了对比,误差分别为 0、0.009 867%。

4.2 振幅-频率调谐参数曲线

参考架于电线杆之间的双线输电线中,高压输电线路的电流约为 500 A 等,考虑磁悬浮列车的速

表1 不同弹簧刚度的 p_1 值

Tab. 1 p_1 with different spring stiffnesses

弹簧刚度 Spring stiffness $k/(\text{N/m})$	0	100	130	3 000	10^9
p_1 值 p_1 value	15.766	15.767	15.768	15.795	10.473

度为 25~150 m/s,在轴向变速运动梁的速度选取时,通常时变轴向速度 c_1 取值一般应在恒定轴向速度 c_0 的 20% 以内。据此选取参数 $i_1 = 500 \text{ A}$, $i_2 = 500 \text{ A}$, $c_0 = 45 \text{ m/s}$, $c_1 = 5 \text{ m/s}$, $P_0 = 3\,000 \text{ N/m}$, $d_1 = 0.04 \text{ m}$, $d_2 = 0.03 \text{ m}$, $F_0 = 3\,000 \text{ N}$, $F_1 = 3\,000 \text{ N}$, $k = 100 \text{ N/m}$, 得到不同参数下幅值-频率调谐参数曲线(图2)。由图2可知,幅频曲线向右弯曲,表现为硬弹簧特性。由图2(a)可见,随着弹簧刚度的增大,共振曲线呈明显的扩张趋势,共振区域明显增加,幅值增大,表明弹簧刚度主要影响系统的振幅。由图2(b)可见,随着电流的增大,曲线左移,说明共振频率发生了改变。图2(c)、图2(d)分别给出不同恒定轴向速度、不同时变轴向速度下幅值-频率调谐参数曲线。由此发现,随着轴向速度的增加,振幅增大且曲线明显左移。由图2(e)、图2(f)可见,随着恒定轴向力的增大,曲线明显左移,振幅增大;随着时变轴向力的增大,曲线明显右移,振幅减小。

4.3 振幅-外激励力(轴向速度)曲线

选取参数 $i_1 = 0$, $i_2 = 8\,000 \text{ A}$, $c_1 = 5 \text{ m/s}$, $d_1 = 0.04 \text{ m}$, $d_2 = 0.03 \text{ m}$, $F_1 = 3\,000 \text{ N}$, $F_0 = 3\,000 \text{ N}$, $c_0 =$

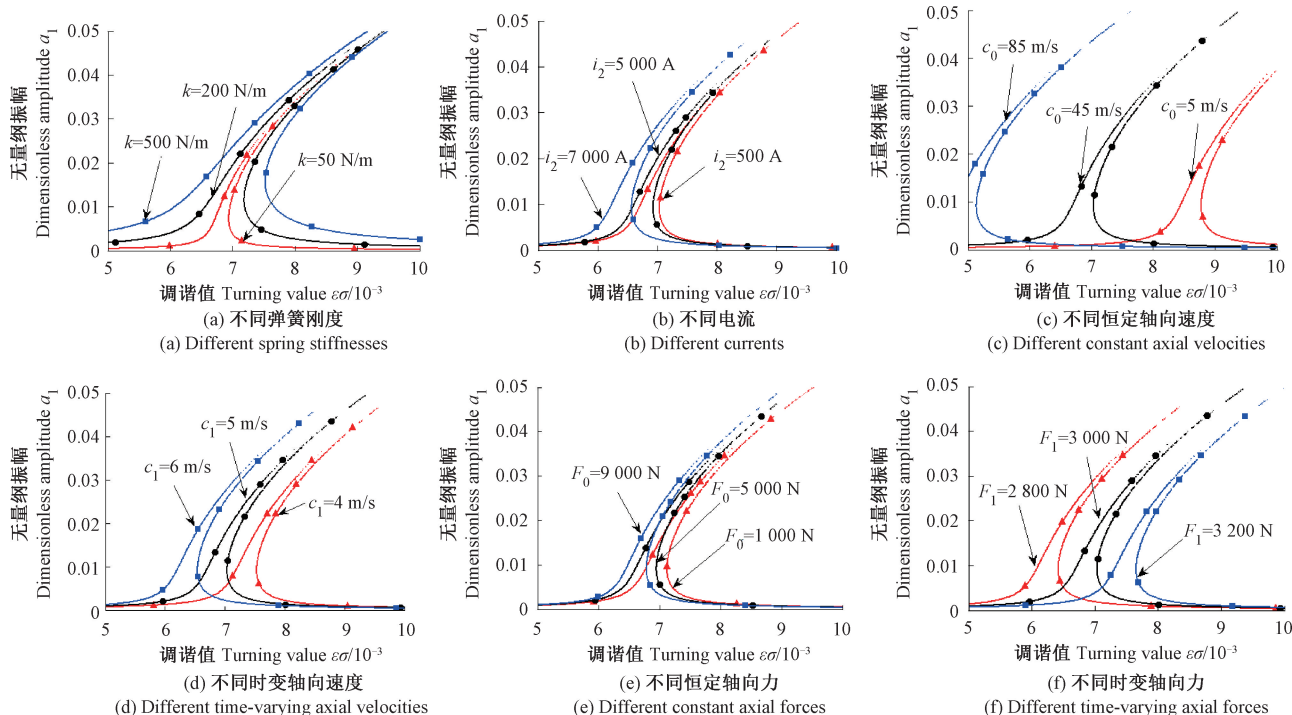


图2 幅值-频率调谐参数曲线

Fig. 2 Curve of amplitude-frequency tuning parameters

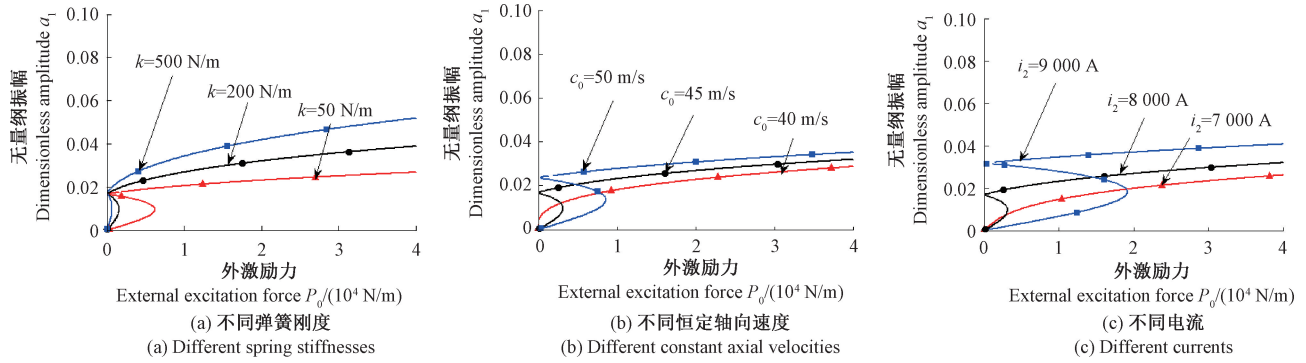


图 3 幅值-外激励力曲线

Fig. 3 Curves of amplitude-external excitation force

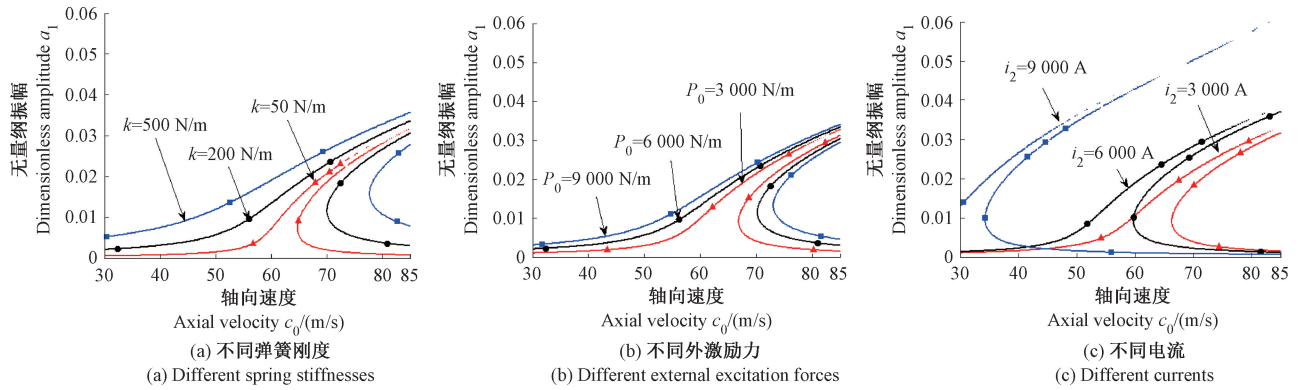
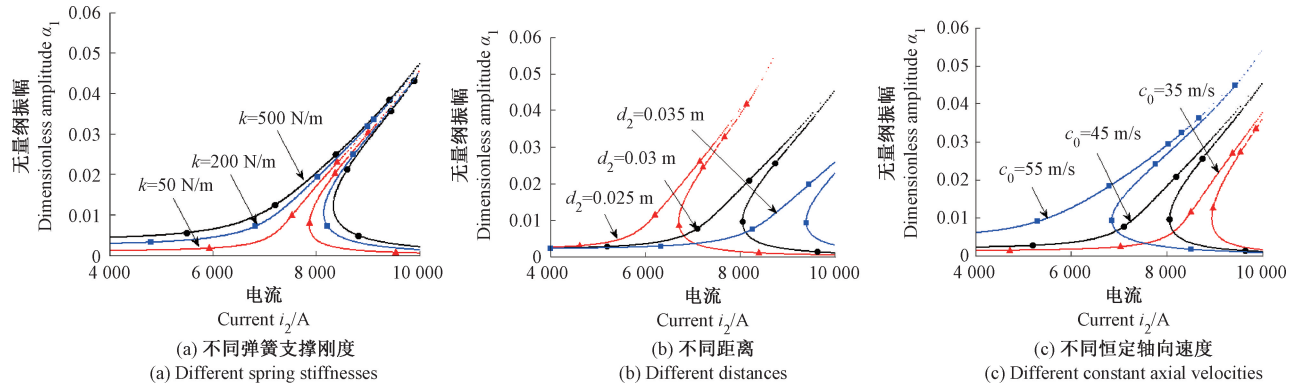
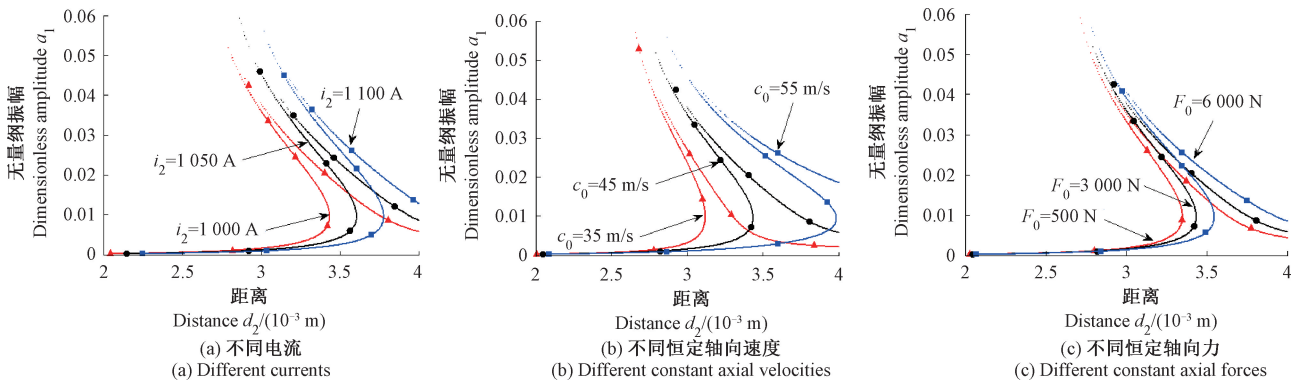


图 4 幅值-轴向速度曲线

Fig. 4 Curves of amplitude-axial velocity

图 5 幅值-电流 i_2 曲线Fig. 5 Curves of amplitude-current i_2 图 6 幅值-距离 d_2 曲线Fig. 6 Curve of amplitude-distance d_2

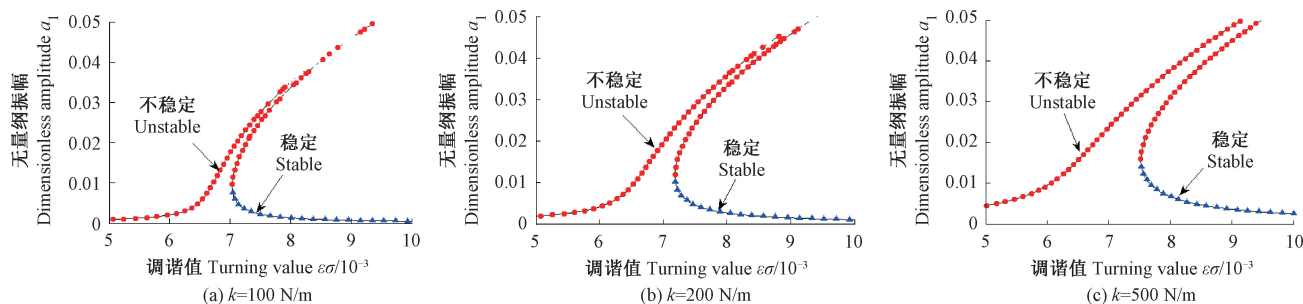


图 7 不同弹簧刚度下幅频曲线的稳定性分析

Fig. 7 Stability analysis of amplitude frequency curve with different spring stiffnesses

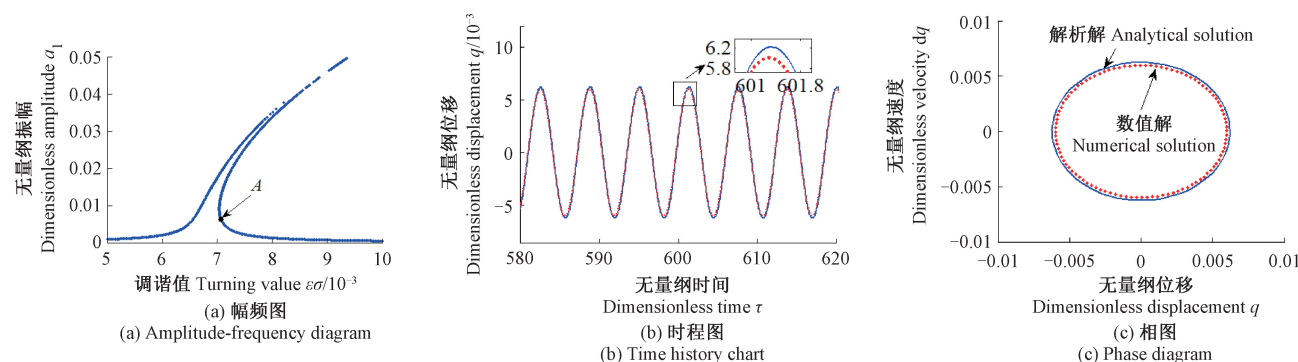


图 8 解析解与数值解对比

Fig. 8 Comparison of analytical solutions and numerical solutions

45 m/s, $k = 100$ N/m, $\varepsilon\sigma = 0.006$, 分别改变 k 、 c_0 、 i_2 等参数的取值, 得到不同参数下的幅值-外激励力曲线(图 3)。由图 3 可以看出, 随着轴向速度和电流的增加, 曲线明显右移。随着弹簧刚度的增加, 曲线呈扩张趋势, 表明电流、轴向速度的改变对共振的振幅及频率都有影响, 而弹簧刚度主要对振幅有影响。

选取参数 $i_1 = 100$ A, $i_2 = 1\,000$ A, $c_1 = 5$ m/s, $P_0 = 3\,000$ N/m, $d_1 = 0.04$ m, $F_1 = 3\,000$ N, $F_0 = 3\,000$ N, $c_0 = 45$ m/s, $k = 100$ N/m, $\varepsilon\sigma = 0.006$, 分别改变 k 、 P_0 、 i_2 等参数的取值, 得到不同参数下的幅值-轴向速度曲线。由图 4 可以看出, 随着弹簧刚度和外激励力的增加, 振幅有增大趋势。随着电流的增加, 共振频率和振幅发生了明显改变。

4.4 振幅-导线电流(位置)曲线

选取参数 $i_1 = 100$ A, $c_1 = 5$ m/s, $P_0 = 3\,000$ N/m, $d_1 = 0.04$ m, $d_2 = 0.03$ m, $F_1 = 3\,000$ N, $F_0 = 3\,000$ N, $c_0 = 45$ m/s, $k = 100$ N/m, $\varepsilon\sigma = 0.006$, 分别改变 k 、 d_2 、 c_0 等参数的取值, 得到不同参数下的幅值-电流曲线(图 5)。两条导线同时通电, 导线电流 i_1 值固定不变, i_2 从 0 开始增大, 磁场强度先减小然后反向增大。由图 5 可以看到, 轴向速度的增大, 引起曲线左移且振幅增加; 距离的增大, 引起曲线右移且振幅减小, 说明轴向速度和距离对于改变共振频率的作用相反; 而弹簧刚度的改变主要影响系统振幅的改变。

参考架于电线杆之间的双线输电线中, 高压输电

线路的电流为 1 400 A 等, 选取参数 $i_1 = 1\,000$ A, $i_2 = 1\,000$ A, $c_1 = 5$ m/s, $P_0 = 3\,000$ N/m, $d_1 = 0.04$ m, $F_1 = 3\,000$ N, $F_0 = 3\,000$ N, $c_0 = 45$ m/s, $k = 100$ N/m, $\varepsilon\sigma = 0.006$, 分别改变 i_2 、 c_0 、 F_0 等参数的取值, 得到不同参数下的幅值-距离曲线(图 6)。由图 6 并结合图 5 可以看出, 振幅随距离的变化曲线向左弯曲, 表现为软弹簧特性。由式(3)可知, 电流与距离对产生磁场的作用相反。随着电流、轴向速度和轴向力的增加, 曲线右移。此现象说明, 电流、轴向速度和轴向力对系统的振幅和频率都有影响。

4.5 稳定性分析

选取参数 $i_1 = 1\,000$ A, $i_2 = 1\,000$ A, $c_0 = 45$ m/s, $c_1 = 5$ m/s, $P_0 = 3\,000$ N/m, $F_0 = 3\,000$ N, $F_1 = 3\,000$ N, $d_1 = 0.04$ m, $d_2 = 0.03$ m, 改变弹簧刚度 $k = 100$ N/m, $k = 200$ N/m, $k = 500$ N/m, 分析稳态运动的稳定性(图 7)。图 7 中圆圈标记曲线表示不稳定运动, 三角标记曲线表示稳定运动。随着弹簧刚度的增加, 稳定性区域增大。

4.6 结果对比分析

选取参数 $i_1 = 1\,000$ A, $i_2 = 1\,000$ A, $c_0 = 45$ m/s, $c_1 = 5$ m/s, $P_0 = 3\,000$ N/m, $F_0 = 3\,000$ N, $F_1 = 3\,000$ N, $d_1 = 0.04$ m, $d_2 = 0.03$ m, $\varepsilon\sigma = 0.007\,071$ 。选取图 8(a) 中幅频曲线上的稳定解点 A (0.007 071, 0.006 225) 对应的稳定稳态解 $(a_{10}, \phi_{40}) = (0.006\,25, 0.043\,216)$ 为初始条件, 由式(19)、式(22)求得解析

解,如图8(b)、图8(c)中的实线所示;相同初始条件下利用龙格-库塔数值方法求解式(10)得到数值解,如图8(b)、图8(c)中的虚线所示。通过比较发现,两种方法得到的结果整体上吻合效果良好。

5 结论

本文以两端弹性约束的轴向变速运动梁为研究对象,得到梁的动力学控制方程。基于多尺度法,研究系统在主共振-参激共振联合作用下的非线性动力学行为。通过算例讨论了不同参数对系统稳态运动及稳定性的影响,得到如下结论:

1) 在载流导线产生的磁场中及外激励力作用下,轴向运动梁发生主-参联合共振,弹簧刚度、外激励力主要影响系统的振幅,而轴向速度、电流、轴向力不仅影响振幅,还影响系统的共振频率。

2) 数值解得到的结果与解析果吻合较好。对稳态运动进行稳定性分析发现,随着弹簧刚度的增加,稳定性区域增大。

参考文献 (References)

- [1] 赵雨皓, 杜敬涛, 许得水. 轴向载荷条件下弹性边界约束梁结构振动特性分析[J]. 振动与冲击, 2020, 39(15): 109-117.
ZHAO YuHao, DU JingTao, XU DeShui. Vibration characteristics analysis for an axially loaded beam with elastic boundary restraints [J]. Journal of Vibration and Shock, 2020, 39(15): 109-117 (In Chinese).
- [2] 滕兆春, 昌博, 付小华. 弹性地基上转动功能梯度材料 Timoshenko 梁自由振动的微分变换法求解[J]. 中国机械工程, 2018, 29(10): 1147-1152.
TENG ZhaoChun, CHANG Bo, FU XiaoHua. DTM analysis for free vibrations of rotating functionally graded material Timoshenko beams on elastic foundations[J]. China Mechanical Engineering, 2018, 29(10): 1147-1152 (In Chinese).
- [3] YAU J D, WU Y S, YANG Y B. Impact response of bridges with elastic bearings to moving loads [J]. Journal of Sound and Vibration, 2001, 248(1): 9-30.
- [4] 鲍四元, 周静, 曹津瑞, 等. 端部任意弹性约束变截面地基梁的自由振动特性分析[J]. 应用力学学报, 2020, 37(5): 2228-2234.
BAO SiYuan, ZHOU Jing, CAO JinRui, et al. Free vibration analysis of foundation-beams with variable cross section and arbitrary elastic end-constraints[J]. Chinese Journal of Applied Mechanics, 2020, 37(5): 2228-2234 (In Chinese).
- [5] LIU Y. Dynamic analysis of elastic support beam subject to moving load[J]. Applied Mechanics and Materials, 2012, 256/257/258/259: 918-921.
- [6] 张弛, 毛晓晔, 丁虎, 等. 受轴向激励弹性支承梁的稳定性分析[J]. 动力学与控制学报, 2022, 20(3): 66-76.
ZHANG Chi, MAO XiaoYe, DING Hu, et al. Stability analysis of axially excited beam with elastic boundary[J]. Journal of Dynamics and Control, 2022, 20(3): 66-76 (In Chinese).
- [7] 张登博, 唐有绮, 陈立群. 非齐次边界条件下轴向运动梁的非线性振动[J]. 力学学报, 2019, 51(1): 218-227.
ZHANG DengBo, TANG YouQi, CHEN LiQun. Nonlinear vibrations of axially moving beams with nonhomogeneous boundary conditions [J]. Chinese Journal of Theoretical and Applied Mechanics, 2019, 51(1): 218-227 (In Chinese).
- [8] 李威, 宋志伟. 弹性约束梁参数失稳分析的离散奇异卷积法[J]. 应用力学学报, 2012, 29(6): 623-629.
LI Wei, SONG ZhiWei. Discrete singular convolution method for parametric instability analysis of elastically restrained beams [J]. Chinese Journal of Applied Mechanics, 2012, 29(6): 623-629 (In Chinese).
- [9] ZHANG D B, TANG Y Q, CHEN L Q. Irregular instability boundaries of axially accelerating viscoelastic beams with 1:3 internal resonance [J]. International Journal of Mechanical Sciences, 2017, 133: 535-543.
- [10] 陈明飞, 靳国永, 张艳涛, 等. 弹性约束的功能梯度曲梁等几何振动分析[J]. 振动工程学报, 2020, 33(5): 930-939.
CHEN MingFei, JIN GuoYong, ZHANG YanTao, et al. Isogeometric vibration analysis of functionally graded curved beams with elastic restraints[J]. Journal of Vibration Engineering, 2020, 33(5): 930-939 (In Chinese).
- [11] CHEN L Q, YANG X D. Vibration and stability of an axially moving viscoelastic beam with hybrid supports [J]. European Journal of Mechanics-A Solids, 2006, 25(6): 996-1008.
- [12] HU Y D, WANG J. Principal-internal resonance of an axially moving current-carrying beam in magnetic field [J]. Nonlinear Dynamics, 2017, 90(1): 683-695.
- [13] 胡宇达, 张立保. 轴向运动导电磁梁的磁弹性振动方程[J]. 应用数学和力学, 2015, 36(1): 70-77.
HU YuDa, ZHANG LiBao. Magneto-elastic vibration equations for axially moving conductive and magnetic beams [J]. Applied Mathematics and Mechanics, 2015, 36(1): 70-77 (In Chinese).
- [14] WANG J, HU Y D, SU Y, et al. Magneto-elastic internal resonance of an axially moving conductive beam in the magnetic field [J]. Journal of Theoretical and Applied Mechanics, 2019, 57(1): 179-191.
- [15] NAYAK B, DWIVEDY S K, MURTHY K S R K. Dynamic analysis of magnetorheological elastomer-based sandwich beam with conductive skins under various boundary conditions [J]. Journal of Sound and Vibration, 2011, 330(9): 1837-1859.
- [16] PRATHIER B, DWIVEDY S K. Parametric instability of a cantilever beam with magnetic field and periodic axial load [J]. Journal of Sound and Vibration, 2007, 305(4/5): 904-917.
- [17] 郑晓静, 刘信恩. 铁磁导电梁式板在横向均匀磁场中的动力特性分析[J]. 固体力学学报, 2000, 21(3): 243-250.
ZHENG XiaoJing, LIU XinEn. Analysis on dynamic characteristics for ferromagnetic conducting plates in a transverse uniform magnetic field [J]. Acta Mechanica Solida Sinica, 2000, 21(3): 243-250 (In Chinese).
- [18] WANG L, CHEN H H, HE X D. Active H_∞ control of the vibration of an axially moving cantilever beam by magnetic force [J]. Mechanical Systems and Signal Processing, 2011, 25(8): 2863-2878.
- [19] LIU M F, CHANG T P. Vibration analysis of a magneto-elastic beam with general boundary conditions subjected to axial load and external force [J]. Journal of Sound and Vibration, 2005, 288(1/2): 399-411.
- [20] 郭硕鸿. 电动力学[M]. 2版. 北京: 高等教育出版社, 1997: 13-14.
GUO ShuoHong. Electrodynamics [M]. 2nd ed. Beijing: Higher Education Press, 1997: 13-14 (In Chinese).
- [21] NAYFEH A H, MOOK D T. Nonlinear oscillations [M]. New York: John Wiley & Sons, 1995: 56-59.
- [22] 赵政, 周海亭, 陈光治, 等. 机械振动与噪声学[M]. 北京: 科学出版社, 2004: 160-161.
ZHAO Mei, ZHOU HaiTing, CHEN GuangYe, et al. Mechanical vibration and noise science [M]. Beijing: Science Press, 2004: 160-161 (In Chinese).